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Total colorings of planar graphs without intersecting 5-cycles ☆

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ABSTRACT

A total k-coloring of a graph G is a coloring of $V(G) \cup E(G)$ using k colors such that no two adjacent or incident elements receive the same color. The total chromatic number of G is the smallest integer k such that G has a total k-coloring. In this paper, it is proved that if G is a planar graph with maximum degree $\Delta \geq 7$ and without intersecting 5-cycles, that is, every vertex is incident with at most one cycle of length 5, then the total chromatic number of G is $\Delta + 1$.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [1] for the terminology and notation not defined here. Let G be a graph. We use $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ (or simply V, E, Δ and δ) to denote the vertex set, the edge set, the maximum degree and the minimum degree of G, respectively. For a vertex $v \in V$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex* (resp., *k^+-vertex*, *k^--vertex*) is a vertex of degree k (resp., at least k, at most k). A k-cycle is a cycle of length k, and a 3-cycle is usually called a *triangle*. Two cycles are *adjacent* (resp., *intersecting*) if they share at least one edge (resp., vertex).

A *total k-coloring* of a graph G is a coloring of $V \cup E$ using k colors such that no two adjacent or incident elements receive the same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total k-coloring. Clearly, $\chi''(G) \geq \Delta + 1$. For its upper bound, we have the following famous conjecture, which is known as the total coloring conjecture (TCC).

Conjecture A. *For any graph G, $\chi''(G) \leq \Delta + 2$.*

This conjecture was confirmed for general graphs with $\Delta \leq 5$. For its history, readers can see [20]. For planar graphs, the only open case is $\Delta = 6$ (see [11,8]). Interestingly, planar graphs with high maximum degree allow a stronger assertion, that is, that every planar graph with high maximum degree Δ is totally $(\Delta + 1)$-colorable. This result was first established in [2] for $\Delta \geq 14$, and was later extended to $\Delta \geq 9$ (see [9]). However, the values of Δ for which it is still not known whether the assertion holds true are 4, 5, 6, 7 and 8. The study of this has attracted a considerable amount of attention and some neat results on this topic have been obtained, as follows.

Theorem 1. *Let G be a planar graph. Then $\chi''(G) = \Delta + 1$ if one of the following conditions holds:*

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(1) $\Delta \geq 8$ *and G contains no* 5- *or* 6-*cycles with chords (see* [13]*);*
(2) $\Delta \geq 8$ *and for every vertex* $x \in V(G)$*, there is an integer* $k \in \{3, 4, 5, 6, 7, 8\}$ *such that x is incident with at most one cycle of length k (see* [15]*);*
(3) $\Delta \geq 8$ *and for each vertex x, there are two integers* $i, j \in \{3, 4, 5, 6\}$ *such that any two cycles of length i and j which contain x are not adjacent (see* [16]*);*
(4) $\Delta \geq 7$ *and G contains no intersecting* 4-*cycles (see* [10]*);*
(5) $\Delta \geq 7$ *and G contains no intersecting* 3-*cycles (see* [17]*);*
(6) $\Delta \geq 7$ *and every vertex v has an integer* $k_v \in \{3, 4, 5, 6\}$ *such that v is not in any* k_v*-cycle (see* [5]*);*
(7) $\Delta \geq 7$ *and no* 3-*cycle is adjacent to a cycle of length less than* 6 *(see* [18]*);*
(8) $\Delta \geq 6$ *and G contains no* 5-*cycles and* 6-*cycles, or* $\Delta \geq 5$ *and G contains no* 4-*cycles and* 6-*cycles(see* [7]*);*
(9) $\Delta(G) \geq 6$*, G contains no intersecting* 4-*cycles and G contains no intersecting* 3-*cycles, or* 5-*cycles, or* 6-*cycles (see* [14]*);*
(10) $\Delta \geq 6$ *and G contains no* 4-*cycles (see* [12]*);*
(11) $(\Delta, g) \in \{(7, 4), (5, 5), (4, 6), (3, 10)\}$*, where g is the girth of G (see* [4]*);*
(12) $(\Delta, k) \in \{(7, 4), (6, 5), (5, 7), (4, 14)\}$*, where G has no cycle of length from* 4 *to k, where* $k \geq 4$ *(see* [19]*).*

In this paper, we obtain that if G is a planar graph with $\Delta \geq 7$ and without intersecting 5-cycles, then $\chi''(G) = \Delta + 1$. To prove the result, we first establish various structural properties of G. Relying on these properties, we use the discharging method in the detailed proof to obtain a contradiction.

2. The main result and its proof

We will introduce some more notation and definitions here for convenience. Let $G = (V, E, F)$ be a planar graph, where F is the face set of G. The degree of a face f, denoted by $d(f)$, is the number of edges incident with it, where each cut-edge is counted twice. A *k-face* and a *k^+-face* are a face of degree k and a face of degree at least k, respectively. For convenience, a k-face with consecutive vertices $v_1, v_2, \ldots, v_k$ along its boundary in an anticlockwise order is often said to be a $(d(v_1), d(v_2), \ldots, d(v_k))$-face.

Theorem 2. *Let G be a planar graph without intersecting* 5-*cycles. If* $\Delta \geq 7$*, then* $\chi''(G) = \Delta + 1$.

Proof. In [15], Theorem 2 was established for $\Delta \geq 8$. So we assume that $\Delta(G) = 7$. Let G be a minimal counterexample to Theorem 2 in terms of the number of vertices and edges. Then every proper subgraph of G is totally 8-colorable, but G itself is not. So G is 2-connected and the boundary of each face in G is exactly a cycle (i.e. each face cannot pass through a vertex v more than once). We first show some known properties on G.

(a) Every vertex is incident with at most $\left\lfloor \frac{3d(v)}{4} \right\rfloor$ 3-faces.
(b) The subgraph G_{27} of G induced by all edges joining 2-vertices to 7-vertices is a forest (see [2,3]).
For any component G_{27}, we root it at a 7-vertex. In this case, every 2-vertex has exactly one parent and exactly one child, which are 7-vertices.
(c) G contains no edge uv with $\min\{d(u), d(v)\} \leq \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $d(u) + d(v) \leq \Delta + 1$ (see [3]).
(d) G contains no 3-face incident with more than one 4-vertex (see [10]).
(e) If v is a 7-vertex of G with $n_2(v) \geq 1$, then $n_{4^+}(v) \geq 1$ (see [5]).

Lemma 3 *([5]). Suppose v is a* 7*-vertex and* $v_1, v_2, \ldots, v_k$ *are consecutive neighbors of v with* $d(v_1) = d(v_k) = 2$ *and* $d(v_i) \geq 3$ *for* $2 \leq i \leq k-1$*, where* $k \in \{3, 4, 5, 6\}$*. If the face incident with* v, v_i, v_{i+1} *is a* 4*-face* $vv_ix_iv_{i+1}$ *for* $1 \leq i \leq k-1$*, then at least one vertex in* $\{v_2, v_3, \ldots, v_{k-1}\}$ *is a* 4^+*-vertex.*

Lemma 4. *G contains no subgraph isomorphic to one of the configurations in Fig. 1, where the vertices marked by • have no other neighbors in G.*

Proof. The proof that G contains no subgraph isomorphic to one of the configurations in Fig. 1(1)–(6) can be found in [6]. What follows proves that G has no configurations depicted in Fig. 1(7)–(11).

By the minimality of G, $G' = G - vv_7$ has a total 8-coloring φ. Erase the colors on all 3^--vertices. Let $C(v) = \{\phi(uv) : u \in N(v)\} \cup \{\varphi(v)\}$. If $\varphi(v_7x_7) \in C(v)$, then the forbidden colors for vv_7 number at most 7, so vv_7 can be properly colored. By recoloring the erased vertices, we obtain a total 8-coloring of G, a contradiction. So we can assume that $\phi(vv_7) \notin C(v)$. Without loss of generality, assume that $\varphi(v) = 8$, $\varphi(v_7x_7) = 7$, and $\varphi(vv_j) = j$ for $j \in \{1, 2, \ldots, 6\}$. Thus, for each 3^--vertex v_k $(1 \leq k \leq 7)$, there is an edge incident with v_k colored 7; otherwise we can recolor vv_k with 7, and color vv_7 with k to obtain a total 8-coloring of G, a contradiction.

Suppose that G contains a subgraph isomorphic to Fig. 1(7). Then $\varphi(v_iv_{i-1}) = 7$ or $\varphi(v_iv_{i+1})\} = 7$. Without loss of generality, we can assume that $\varphi(v_iv_{i-1}) = 7$. If $\varphi(v_iv_{i+1}) = i - 1$, then we exchange the colors of edges vv_{i+1} and v_iv_{i+1}, vv_{i-1} and $v_{i-1}v_i$, and color vv_7 with $i + 1$. Otherwise, we exchange the colors of edges vv_{i-1} and $v_{i-1}v_i$, and color vv_7 with $i - 1$. By recoloring the erased vertices, we obtain a total 8-coloring of G, a contradiction.

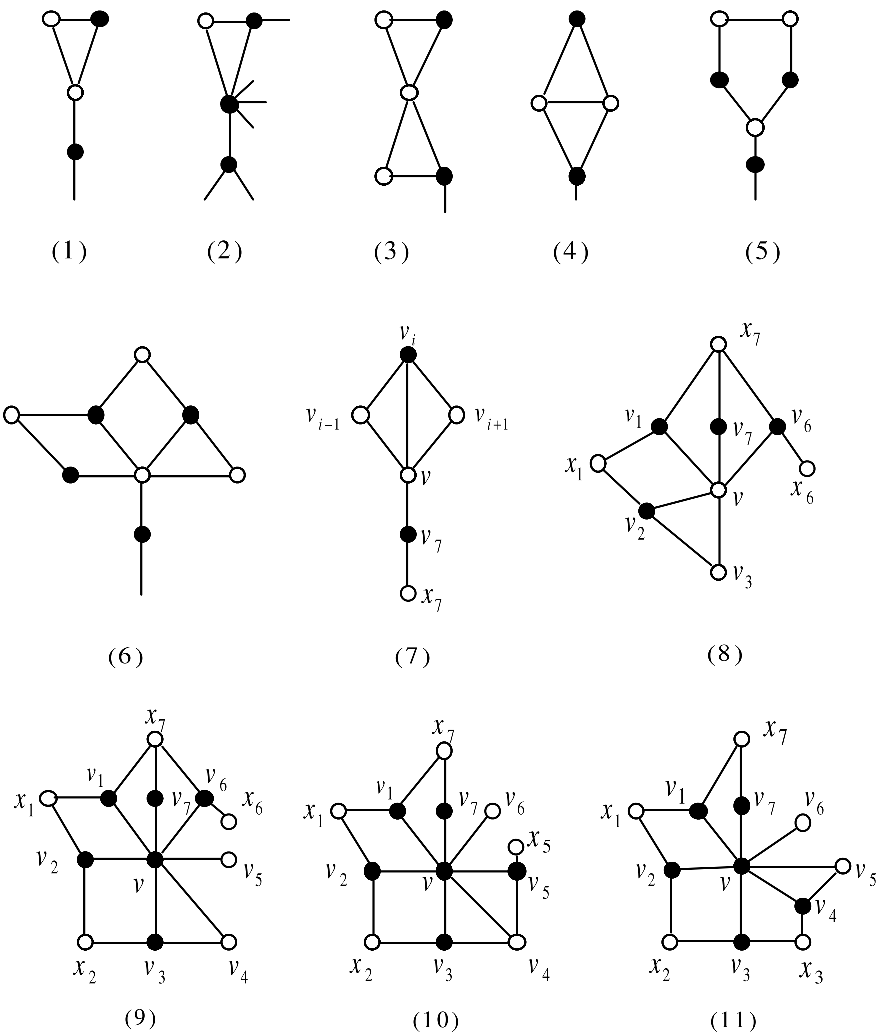

Fig. 1. Reducible configuration.

Suppose that G contains a subgraph isomorphic to Fig. 1(8). We know that $\varphi(v_1x_1) = \varphi(v_6x_6) = \varphi(v_2v_3) = 7$. Suppose that $\varphi(v_1x_7) = 3$. We exchange the colors of edges x_7v_1 and x_7v_6. If $\varphi(v_6x_7) = 1$, then we additionally exchange the colors of edges vv_1 and vv_6. So we can assume that $\varphi(v_1x_7) \neq 3$. Thus, we first exchange the colors of vv_3 and v_2v_3, and color vv_7 with 3. Then, if $\varphi(v_2x_1) = 3$, then we exchange the colors of x_1v_2 and x_1v_1. Finally, we recolor the erased vertices; we obtain a total 8-coloring of G, a contradiction.

Suppose that G contains a subgraph isomorphic to Fig. 1(9). By the same argument as above, we assume that $\varphi(v_1x_1) = \varphi(v_2x_2) = \varphi(v_3v_4) = \varphi(v_6x_6) = 7$, $\varphi(x_2v_3) = \varphi(x_1v_2) = \varphi(v_1x_7) = 4$ and $\varphi(v_1x_7) \neq 4$. Now we exchange the colors of edges vv_4 and v_3v_4, v_3x_2 and v_2x_2, v_2x_1 and x_1v_1 to obtain a total 8-coloring, a contradiction.

Suppose that G contains a subgraph isomorphic to Fig. 1(10). By the same argument as above, we assume that $\varphi(v_3v_4) = \varphi(v_5x_5) = 7$ and $\varphi(x_2v_3) = 4$. First, if $\varphi(v_4v_5) = 3$, then we exchange the colors of edges vv_3 and vv_5. Then we recolor v_3v_4 with 3, v_4v_5 with 4, and vv_4 with 7. Finally, we color vv_7 with 4 and recolor the erased vertices. We obtain a total 8-coloring of G, a contradiction.

Suppose that G contains a subgraph isomorphic to Fig. 1(11). By the same argument as above, we assume that $\varphi(v_1x_1) = \varphi(v_2x_2) = \varphi(v_3x_3) = \varphi(v_4v_5) = 7$ and $\varphi(x_3v_4) = \varphi(x_2v_3) = \varphi(x_1v_2) = \varphi(v_1x_7) = 5$. If $\varphi(v_6) = 7$, then we exchange the colors of edges vv_5 and v_4v_5, v_4x_3 and v_3x_3, v_3x_2 and x_2v_2, x_1v_2 and x_1v_1, v_1x_7 and x_7v_7 and recolor v with 5 and color vv_7 with 8. Otherwise, we recolor v with 7 and color vv_7 with 8. By recoloring the erased vertices, we obtain a total 8-coloring of G, a contradiction. $\square$

Since G is a planar graph, by Euler's formula, we have

$$\sum_{v\in V}(2d(v) - 6) + \sum_{f\in F}(d(f) - 6) = -12 < 0.$$

Now we define the initial charge function $ch(x)$ of $x \in V \cup F$ to be $ch(v) = 2d(v) - 6$ if $v \in V$ and $ch(f) = d(f) - 6$ if $f \in F$. It follows that $\sum_{x \in V \cup F} ch(x) < 0$. Now we design appropriate discharging rules and redistribute weights accordingly. Note that any discharging procedure preserves the total charge of G. If we can define suitable discharging rules to charge the initial charge function ch to the final charge function ch' on $V \cup F$, such that $ch'(x) \geq 0$ for all $x \in V \cup F$, then we get an obvious contradiction.

Our discharging rules are defined as follows.

R1. Let v be a 2-vertex. If v is incident with a 3-face, then it receives 1 from each of its neighbors. Otherwise, v receives $\frac{3}{2}$ from its child and $\frac{1}{2}$ from its parent.

R2. Let f be a 3-face. If f is incident with a 3^--vertex, then it receives $\frac{3}{2}$ from each of its incident 6^+-vertices. If f is incident with a 4-vertex, then it receives $\frac{1}{2}$ from the 4-vertex and receives $\frac{5}{4}$ from each of its incident 5^+-vertices. If f is not incident with any 4^--vertex, then it receives 1 from each of its incident 5^+-vertices.

R3. Let f be a 4-face. If f is incident with two 3^--vertices, then it receives 1 from each of its two incident 6^+-vertices. If f is incident with the unique 3^--vertex u, then it receives $\frac{3}{4}$ from each of its two incident 6^+-vertices adjacent to u and receives $\frac{1}{2}$ from another 4^+-vertex. If f is incident with no 3^--vertices, then it receives $\frac{1}{2}$ from each of its incident vertices.

R4. Let f be a 5-face. If f is incident with two 3^--vertices, then it receives $\frac{1}{3}$ from each of its incident 4^+-vertices. If f is incident with one 3^--vertex, then it receives $\frac{1}{4}$ from each of its incident 4^+-vertices. If f is not incident with any 3^--vertex, then it receives $\frac{1}{5}$ from each of its incident vertices.

In the rest of this paper we check that $ch'(x) \geq 0$ for all $x \in V \cup F$. Firstly note that our discharging rules are just designed such that $ch'(f) \geq 0$ for all $f \in F$ and $ch'(v) \geq 0$ for all 2-vertices $v \in V$. So we only check that $ch'(v) \geq 0$ for all 3^+-vertices G.

Let v be a vertex of G. If $d(v) = 3$, then $ch'(v) = ch(v) = 0$. If $d(v) = 4$, then v sends at most $\frac{1}{2}$ to each of its incident faces by R2 and R3, and it follows that $ch'(v) \geq ch(v) - \frac{1}{2} \times 4 = 0$.

Suppose $d(v) = d \geq 5$. Let $n_t(v)$ be the number of t-vertices adjacent to a vertex v, and $f_k(v)$ the number of k-faces incident with v. In particular, denote $f_3(v)$ by t. Let $v_1, v_2, \ldots, v_d$ be neighbors of v and $f_1, f_2, \ldots, f_d$ be faces incident with v in an anticlockwise order, where f_i is incident with v_i and v_{i+1}, for all i such that $i \in \{1, 2, \ldots, d\}$. Note that all the subscripts in the paper are taken modulo d. By the choice of G, We have the following observations.

(O_1) Let f_i be a 3-face and f_{i-1} (or f_{i+1}) be a 4-face. If f_i is not incident with a 2-vertex, then f_{i+1} (or f_{i-1}) must be a 5^+-face.

(O_2) If $d(v_i) = 2$, $d(f_{i-2}) = 3$ and $4 \leq d(f_{i-1}) \leq 5$, then $d(f_i) \geq 6$.

(O_3) Let $d(v_i) = 2$, $d(f_{i-1}) = 3$ and $d(f_i) = 4$. If $d(f_{i-2}) = 5$, then $d(f_{i+1}) \geq 6$; if $d(f_{i-2}) = 3$ and $d(f_{i-3}) = 4$ or 5, then $d(f_{i+1}) \geq 6$; if $d(f_{i-3}) = 3$, then $d(f_{i-2}) \geq 5$. Furthermore, if $d(f_{i-2}) = 5$, then $d(f_{i+1}) \geq 6$.

Suppose $d(v) = 5$. Then v is incident with at most three 3-faces, that is, $t \leq 3$ by (a). If $t = 3$, then $f_{6^+}(v) + f_5(v) \geq 2$. Moreover, if some f_i is a 5-face, then f_i is incident with at most one 3^--vertex by (c). So $ch'(v) \geq ch(v) - \frac{5}{4} \times 3 - \frac{1}{4} = 0$ by R2 and R4. If $1 \leq t \leq 2$, then $f_{6^+}(v) \geq 1$ by (O_1), and it follows that $ch'(v) \geq ch(v) - \frac{5}{4} \times t - \frac{1}{2} \times (5 - 1 - t) = \frac{8-3t}{4} > 0$ by R2 and R3. If $t = 0$, then v is incident with five 4^+-faces, and we have $ch'(v) \geq ch(v) - \frac{1}{2} \times 5 > 0$ by R3.

Suppose $d(v) = 6$. Then $t \leq 4$ by (a). If $t = 0$, then $ch'(v) \geq ch(v) - 6 \times 1 = 0$. If $1 \leq t \leq 2$, then $f_{6^+}(v) \geq 1$ by (O_1), and it follows that $ch'(v) \geq ch(v) - \frac{3}{2} \times t - (6 - 1 - t) = \frac{2-t}{2} \geq 0$. Suppose that $3 \leq t \leq 4$. Then $f_{6^+}(v) \geq 1$ and $f_{5^+}(v) \leq 1$. By Lemma 4(2), if some f_i is a 3-face and incident with a 3-vertex, then all neighbors except the 3-vertex of v are 4^+-vertices. Note that v can be incident with two adjacent $(3, 6^+, 6^+)$-faces, v sends at most $\frac{3}{2}$ to a $(3, 6^+, 6^+)$-face and at most $\frac{5}{4}$ to a $(4, 5^+, 5^+)$-face by R2, at most 1 to a 4^+-face by R3 and at most $\frac{1}{3}$ to a 5^+-face by R4. So $ch'(v) \geq ch(v) - \frac{3}{2} \times 2 - \frac{5}{4} \times (t - 2) - \frac{1}{3} - (6 - 1 - 1 - t) = \frac{14-3t}{12} > 0$.

Suppose $d(v) = 7$. Then $ch(v) = 2 \times 7 - 6 = 8$. If $n_2(v) \geq 1$ and any 2-vertex is not incident with a 3-face, then v sends at most $\frac{n_2(v)+2}{2}$ to all its adjacent 2-vertices by R1.

Lemma 5. *Suppose that $d(v_i) = d(v_k) = 2$ and $d(v_j) \geq 3$ for all $j = i + 1, \ldots, k - 1$. If $f_i, f_{i+1}, \ldots, f_{k-1}$ are 4^+-faces, then v sends at most $\frac{3}{2} + (k - i - 2)$ (in total) to $f_i, f_{i+1}, \ldots, f_{k-1}$.*

Proof. By Lemma 3, $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \geq 4$ or $\max\{d(f_i), \ldots, d(f_{k-1})\} \geq 5$. If $\max\{d(v_{i+1}), \ldots, d(v_{k-1})\} \geq 4$, then v sends at most $2 \times \frac{3}{4} + (k - i - 2)$ to $f_i, \ldots, f_{k-1}$ by R3. If $\max\{d(f_i), \ldots, d(f_{k-1})\} \geq 5$, then v sends at most $\frac{1}{3} + (k - i - 1)$ to $f_i, \ldots, f_{k-1}$ by R3 and R4. Since $2 \times \frac{3}{4} > 1 + \frac{1}{3}$, v sends at most $\frac{3}{2} + (k - i - 2)$ to $f_i, f_{i+1}, \ldots, f_{k-1}$. □

We consider the following cases.

Case 1. $n_2(v) \geq 6$. Then $f_{6^+}(v) \geq 5$ and $f_3(v) = 0$ by Lemma 4(1,5). So $ch'(v) \geq ch(v) - \frac{6+2}{2} - 2 > 0$.

Case 2. $n_2(v) = 5$. If $t = 0$, then $f_{6^+}(v) \geq 3$, and it follows that $ch'(v) \geq ch(v) - \frac{5+2}{2} - 4 \times 1 > 0$. Otherwise, $f_{6^+}(v) \geq 4$. So $ch'(v) \geq ch(v) - \frac{5+2}{2} - \frac{3}{2} - 2 \times 1 > 0$.

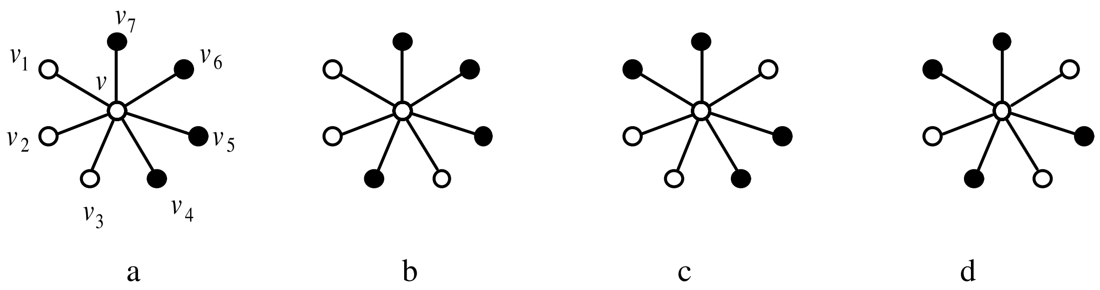

Fig. 2. $n_2(v) = 4$.

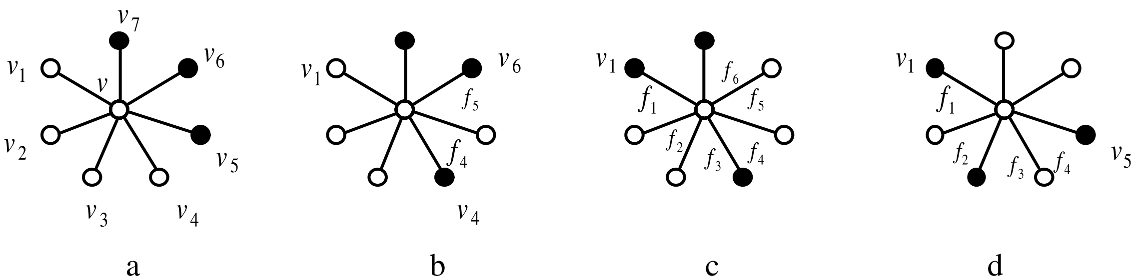

Fig. 3. $n_2(v) = 3$.

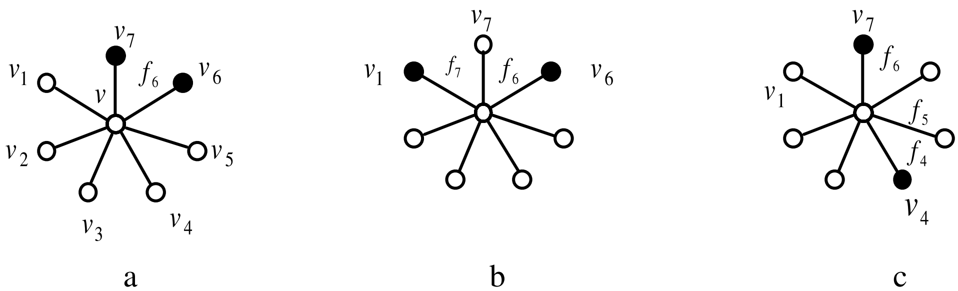

Fig. 4. $n_2(v) = 2$.

Case 3. $n_2(v) = 4$. All 2-vertices incident with v are located as shown in Fig. 2, where the vertices marked by $\bullet$ are 2-vertices.

For Fig. 2(a), $t \le 2$ by Lemma 4(1). By Lemma 4(5), f_4, f_5, f_6 are 6^+-faces. So $f_{6^+}(v) \ge 3$. If $1 \le t \le 2$, then $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} \times t - (7 - 3 - t) = \frac{2-t}{2} \ge 0$. Otherwise, $ch'(v) \ge ch(v) - \frac{4+2}{2} - \left(\frac{3}{2} + 2\right) > 0$ by Lemma 5. For Fig. 2(b) and (c), $t \le 1$. If $t = 1$, then $f_{6^+}(v) \ge 3$ by Lemma 4(5) and (O_1), and it follows that $ch'(v) \ge ch(v) - \frac{4+2}{2} - \frac{3}{2} - (7 - 3 - 1) > 0$. Otherwise, $ch'(v) \ge ch(v) - \frac{4+2}{2} - \left(\frac{3}{2}\right) - \left(\frac{3}{2} + 1\right) > 0$ by Lemma 5. For Fig. 2(d), we have $t = 0$ and $ch'(v) \ge ch(v) - \frac{4+2}{2}3 \times \left(\frac{3}{2}\right) > 0$ by Lemma 5.

Case 4. $n_2(v) = 3$. All 2-vertices incident with v are located as shown in Fig. 3, where the vertices marked by $\bullet$ are 2-vertices.

For Fig. 3(a), $t \le 3$ by Lemma 4(1). If $1 \le t \le 3$, then $f_{6^+}(v) \ge 3$, and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} \times t - (7 - 3 - t) = \frac{3-t}{2} \ge 0$. Otherwise $ch'(v) \ge ch(v) - \frac{3+2}{2} - \left(\frac{3}{2} + 3\right) > 0$. For Fig. 3(b), $t \le 2$. Then v sends at most $\frac{3}{2}$ to f_4 and f_5 by Lemma 5. If $1 \le t \le 2$, then $f_{6^+}(v) \ge 2$ and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} \times t - \left(\frac{3}{2}\right) - (7 - 2 - 2 - t) = \frac{2-t}{2} \ge 0$. Otherwise $ch'(v) \ge ch(v) - \frac{3+2}{2} - \left(\frac{3}{2} + 2\right) - \frac{3}{2} > 0$. For Fig. 3(c), $t \le 2$. If $t = 2$, then $f_{6^+}(v) \ge 4$ and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} \times 2 - 1 > 0$. If $t = 1$, then $f_{6^+}(v) \ge 2$ and it follows that $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - \left(\frac{3}{2} + 1\right) - 1 > 0$. If $t = 0$, then $ch'(v) \ge ch(v) - \frac{3+2}{2} - \left(\frac{3}{2} + 1\right) \times 2 > 0$. For Fig. 3(d), $t \le 1$. Then v sends at most $2 \times \frac{3}{2}$ to f_1, f_2, f_3 and f_4 by Lemma 5. If $t = 1$, then the 3-face incident with v must be f_6, and $d(f_5) \ge 6$ or $d(f_7) \ge 6$ by (O_1) and we have $ch'(v) \ge ch(v) - \frac{3+2}{2} - \frac{3}{2} - \left(\frac{3}{2}\right) \times 2 - 1 \ge 0$. Otherwise, $ch'(v) \ge ch(v) - \frac{3+2}{2} - \left(\frac{3}{2} + 1\right) - 2 \times \left(\frac{3}{2}\right) = 0$.

Case 5. $n_2(v) = 2$. All 2-vertices incident with v are located as shown in Fig. 4, where the vertices marked by $\bullet$ are 2-vertices.

For Fig. 4(a), $t \le 3$. If $t = 3$, then $f_{6^+}(v) \ge 3$ by Lemma 4(5) and O_1, and we have $ch'(v) \ge ch(v) - 2 - \frac{3}{2} \times 3 - 1 > 0$. If $1 \le t \le 2$, then $f_{6^+}(v) \ge 2$ and it follows that $ch'(v) \ge ch(v) - 2 - \frac{3}{2} \times t - (7 - 2 - t) = \frac{2-t}{2} \ge 0$. If $t = 0$, then $ch'(v) \ge ch(v) - \left(\frac{3}{2} + 4\right) > 0$ by Lemma 5.

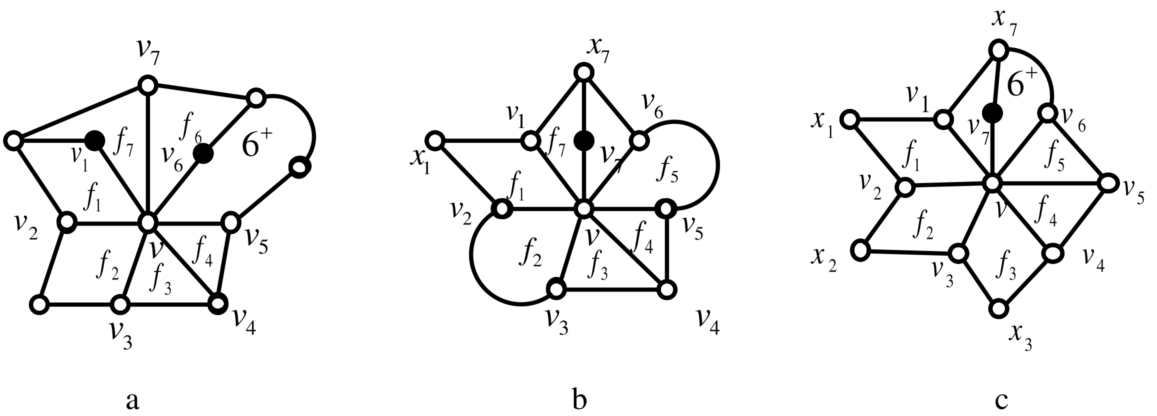

Fig. 5. Three special graphs.

For Fig. 4(b), $t \le 3$. If $t = 3$, that is, f_2, f_3, f_4 are 3-faces, then f_1, f_5 are 6^+-faces. It follows that $ch'(v) \ge ch(v) - 2 - \frac{3}{2} \times 3 - \frac{3}{2} = 0$, where v sends at most $\frac{3}{2}$ to f_6 and f_7 by Lemma 5. Suppose that $t = 2$. If v is incident with at least two 6^+-faces, then $ch'(v) \ge ch(v) - \frac{2+2}{2} - \frac{3}{2} \times 2 - 3 = 0$. Otherwise, if v is incident with one 6^+-face and one 5-face, then $ch'(v) \ge ch(v) - \frac{2+2}{2} - \frac{3}{2} \times 2 - \frac{1}{3} - \left(\frac{3}{2}\right) - 1 = \frac{1}{6} > 0$, where v sends at most $\frac{3}{2}$ to f_6 and f_7 by Lemma 5. Otherwise, two 3-faces incident with v must be adjacent. Without loss of generality, assume that f_3 and f_4 are the 3-faces. By the choice of G, it follows that f_5 is a 6^+-face, and f_1, f_2, f_6, f_7 are 4-faces (see Fig. 5(a)). By Lemma 4(7), $d(v_4) \ge 4$. If $d(v_2) \ge 4$, then v sends at most $\frac{3}{4}$ to f_1 and f_2, respectively, and it follows that $ch'(v) \ge ch(v) - 2 - \frac{3}{4} \times 2 - \frac{3}{2} \times 2 - \frac{3}{2} = 0$, where v sends at most $\frac{3}{2}$ to f_6 and f_7 by Lemma 5. Otherwise $d(v_2) = 3$ and $d(v_3) \ge 4$ by Lemma 4(6). So $ch'(v) \ge ch(v) - 2 - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 1 - \frac{3}{2} = 0$, where v sends at most $\frac{5}{4}$ to f_3 by R2, and at most $\frac{3}{4}$ to f_2 by R3. If $t = 1$, then $f_{6^+}(v) \ge 1$ by (O_1), and it follows that $ch'(v) \ge ch(v) - \frac{2+2}{2} - \frac{3}{2} - 3 - \left(\frac{3}{2}\right) = 0$. If $t = 0$, then $ch'(v) \ge ch(v) - \frac{2+2}{2} - \frac{3}{2} - \left(\frac{3}{2} + 3\right) = 0$.

For Fig. 4(c), $t \le 3$ by Lemma 4(1). If $t = 3$, that is, f_1, f_2, f_5 are 3-faces, then v is incident with at least three 6^+-faces by (O_1), and it follows that $ch'(v) \ge ch(v) - \frac{2+2}{2} - \frac{3}{2} \times 3 - 1 > 0$. If $1 \le t \le 2$, then v is incident with at least two 6^+-faces, or one 6^+-face and one 5^+-face by (O_1) and by (O_2). So $ch'(v) \ge ch(v) - 2 - \frac{3}{2} \times t - \max\left\{7 - 2 - t, \frac{1}{3} + \left(\frac{3}{2} + 1\right) + (7 - 2 - 3 - t)\right\} = 0$. If $t = 0$, then $ch'(v) \ge ch(v) - 2 - \frac{3}{2} - \left(\frac{3}{2} + 1\right) - \left(\frac{3}{2} + 2\right) > 0$.

Case 6. $n_2(v) = 1$. Without loss of generality, assume that $d(v_7) = 2$ and f_7 is a 4^+-face. By (e), we have $n_{4^+}(v) \ge 1$.

Case 6.1. v_7 is incident with a 3-face, that is, f_6 is a $(2, 7, 7)$-face. Then v sends at most 1 to 2-vertex v_7 by R1. Note that $t \le 5$ and all other 3-faces except f_6 incident with v are $(4^+, 5^+, 7)$-faces by Lemma 4(3,4). If $4 \le t \le 5$, then v is incident with at least two 6^+-faces and at most one 5^+-faces. So $ch'(v) \ge ch(v) - 1 - \frac{3}{2} - \frac{5}{4} \times (t-1) - \frac{1}{3} - (7-2-2-t) > \frac{5-t}{4} > 0$. If $1 \le t \le 3$, then v is incident with at least one 6^+-face by (O_3), and it follows that $ch'(v) \ge ch(v) - 1 - \frac{3}{2} - \frac{5}{4} \times (t-1) - (7-1-t) = \frac{3-t}{4} \ge 0$.

Case 6.2. v_7 is not incident with any 3-face. Then $t \le 4$. If $t = 0$, then $ch'(v) \ge ch(v) - \frac{3}{2} - 2 \times \frac{3}{4} - 5 \times 1 = 0$ by R3. If $t = 1$, then $f_{6^+}(v) \ge 1$ by (O_1), and it follows that $ch'(v) \ge ch(v) - \frac{3}{2} - \frac{3}{2} - 1 \times 5 = 0$. If $t = 3$, then $f_{6^+}(v) \ge 2$ and it follows that $ch'(v) \ge ch(v) - \frac{3}{2} - 3 \times \frac{3}{2} - 2 = 0$. If $t = 4$, then $f_{6^+}(v) \ge 2$ and $f_5(v) \le 1$. So $ch'(v) \ge ch(v) - \frac{3}{2} - 4 \times \frac{3}{2} - \frac{1}{3} > 0$. In the following, we assume that $t = 2$.

If v is incident with at least two 6^+-faces or v is incident with one 6^+-face and one 5-face, then $ch'(v) \ge ch(v) - \frac{3}{2} - \frac{3}{2} \times 2 - \frac{1}{3} - 3 > 0$. Otherwise, two 3-faces incident with v must be adjacent and v is incident with only one 6^+-face and four 4^+-faces, where the unique 6^+-face is adjacent to one of the two 3-faces. Without loss of generality, assume that f_i and f_{i+1} are the 3-faces, where $i \in \{3, 4\}$ (see Fig. 5(b) and (c)). By Lemma 4(7), we have $d(v_{i+1}) \ge 4$. If $d(v_{i+1}) = 4 or 5$. Then v is incident with two $(4^+, 5^+, 7)$-faces by (c), and it follows that $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{5}{4} \times 2 - 4 = 0$ by R2. Otherwise $d(v_{i+1}) \ge 6$.

For Fig. 5(b), $i = 3$. We have $d(f_2) = 4$ and $d(f_5) = 6$, or $d(f_2) = 6$ and $d(f_5) = 4$. Suppose $d(f_2) = 4$. If $d(v_3) \ge 4$, then $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 3 = 0$ by R2 and R3. If $d(v_1) \ge 4$ or $d(v_2) \ge 4$ then $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} \times 2 - \frac{3}{4} \times 2 - 2 = 0$ by R3. Otherwise, $d(v_1) = d(v_2) = d(v_3) = 3$; then we have $d(v_5) \ge 4$ and $d(v_6) \ge 4$ by Lemma 4(9) and (10), and it follows that $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 3 = 0$. Assume that $d(f_5) = 4$ and $d(f_2) \ge 6$. If $d(v_1) \ge 4$ or $d(v_6) \ge 4$, then $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} \times 2 - \frac{3}{4} \times 2 - 2 = 0$ by R3. Otherwise, $d(v_5) \ge 4$ by Lemma 4(8), and it follows that $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 3 = 0$ by R2 and R3.

For Fig. 5(c), $i = 4$. We have $d(f_6) \ge 6$. If $d(v_i) \ge 4$ for $i \in (1, 2, 3)$, then $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} \times 2 - \frac{3}{4} \times 2 - 2 = 0$ by R3. Otherwise, $d(v_1) = d(v_2) = d(v_3) = 3$; then we have $d(v_4) \ge 4$ by Lemma 4(11), and then $ch'(v) \ge ch(v) - \frac{1+2}{2} - \frac{3}{2} - \frac{5}{4} - \frac{3}{4} - 3 = 0$ by R2 and R3.

Case 7. $n_2(v) = 0$. Note that $t \leq 5$ by (a). If $t = 5$, then $f_{6^+}(v) \geq 2$, and it follows that $ch'(v) \geq ch(v) - \frac{3}{2} \times 5 > 0$. If $1 \leq t \leq 4$, then $f_{6^+}(v) \geq 1$ by (O_1), and it follows that $ch'(v) \geq ch(v) - \frac{3}{2} \times t - (7 - 1 - t) = \frac{4-t}{2} \geq 0$. Otherwise $ch'(v) \geq ch(v) - 7 \times 1 > 0$. Hence we complete the proof of the theorem. □

3. Conclusions

Combining Theorem 2 and the results (2) and (4)–(6) of Theorem 1, we are sure that the following result holds.

Conjecture 6. *Let G be a planar graph. If $\Delta \geq 7$ and every vertex v is incident with at most one i-cycle for some $i \in \{3, 4, 5, 6\}$, then $\chi''(G) = \Delta + 1$.*

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